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Field Theoretical Quantum Effects on the Kerr Geometry

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Abstract

We study quantum aspects of the Einstein gravity with one time-like and one space-like Killing vector commuting with each other. The theory is formulated as a $SL(2,\mathbf{R})/U(1)$ nonlinear σ-model coupled to gravity. The quantum analysis of the nonlinear σ-model part, which includes all the dynamical degrees of freedom, can be carried out in a parallel way to ordinary nonlinear σ-models in spite of the existence of an unusual coupling. This means that we can investigate consistently the quantum properties of the Einstein gravity, though we are limited to the fluctuations depending only on two coordinates. We find the forms of the beta functions to all orders up to numerical coefficients. Finally we consider the quantum effects of the renormalization on the Kerr black hole as an example. It turns out that the asymptotically flat region remains intact and stable, while, in a certain approximation, it is shown that the inner geometry changes considerably however small the quantum effects may be.

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1 Introduction

Of quantum aspects of Einstein gravity we are far from a clear understanding at present. There exist many difficulties both conceptually and technically. One of the most outstanding ones is its nonrenormalizability. Due to this, we have no consistent way to investigate its quantum theory. In spite of the difficulties, many attempts have been made for incorporating quantum effects, for example, by using semiclassical and $1/N$ approximations. In these approaches, the flat space-time is suffered from the instability by the quantum perturbation owing to the induced higher derivative terms or the tachyonic modes in the gravitational propagator [1]-[4]. Moreover, the tachyonic modes make the actual perturbative calculation impossible. The theories with higher derivative terms are studied also as an effective ones in the low energy limit of some fundamental theory such as string theory (see, e.g., [5]). In these theories, the higher derivative terms appear in the perturbation with respect to weak curvature. Hence we cannot deal with the region with strong curvature where quantum effects are expected to become important, and we can consider only small deviations from the classical solutions. There is also ambiguity related to field redefinitions ([6], for instance). In a much more simplified setting, the quantum mechanics of minisuperspace [7] or the Schwarzschild black hole has been investigated [8]. In these cases, it is still difficult to extract their physical consequences. Thus we have not yet succeeded in grasping definite quantum aspects of general relativity even in some approximation.

Difficulties concerning quantum Einstein gravity are expected to be overcome when we understand a more fundamental theory. Intensive studies have been made in this direction, but, together with this, it may be important to accumulate certain pieces of knowledge of quantum properties of Einstein gravity even if in a simplified setting.

In this article, we shall work with the Einstein gravity with one time-like and one space-like Killing vector commuting with each other. The Einstein gravity with two commuting Killing vectors can be formulated as a $SL(2,\mathbf{R})/U(1)$ nonlinear σ-model coupled to gravity [9, 10]. For the Einstein-Maxwell system, we have a similar formulation as a $SU(2,1)/SU(2)\times U(1)$ or $SU(2,1)/SU(1,1)\times U(1)$ nonlinear σ-model according to the signatures of the Killing vectors. One of the most important applications of these facts is the proof of the uniqueness theorem of the Kerr-Newman solution by [11]. The central equation of these systems is known as the Ernst equation [12]. For generating the exact solutions, these systems have been studied extensively and many interesting and

rich structures have been revealed [9, 10], [13]. In particular, the systems possess infinite dimensional hidden symmetries [14]-[16] and become integrable [17, 18]. In addition, the similarity between these hidden symmetries and those of dimensionally reduced supergravities has been recognized [19], and recently applied to the study of string dualities [20].

As for nonlinear σ-models, there exists an extensive literature on its quantum analysis [21]-[24]. In two dimensions, nonlinear σ-models are renormalizable (in a generalized sense by Friedan [22]) and their quantum aspects can be studied in a consistent way at least perturbatively. Furthermore, among various models, the simplest one is the $O(3)$ or $\mathbf{C}P^1(SU(2)/U(1))$ nonlinear σ-model, and its target manifold $\mathbf{C}P^1$ is the compact analog of our $SL(2,\mathbf{R})/U(1)$.

Therefore we can expect to make use of the vast knowledge in the literatures. The purpose of this article is to study the quantum theory of the Einstein gravity reduced to two dimensions and to investigate its effects on geometry. Of course, in our formulation in which some of the quantum fluctuations are truncated, we can say only a little about the statistical aspects of the original Einstein gravity. However it turns out that we can actually deal with the quantum theory of this reduced Einstein gravity and evaluate some effects on geometry in a consistent and simple way. We believe that our analysis gives some insights into quantum aspects of Einstein gravity.

The rest of this article is organized as follows. In Sec.2, we formulate the Einstein gravity with the two Killing vectors as a $SL(2,\mathbf{R})/U(1)$ nonlinear σ-model and equations governing the system are derived. Next, in Sec.3, we investigate the quantum theory of the nonlinear σ-model part. The beta functions are obtained to all orders up to numerical coefficients determined by explicit loop calculations. Then the equations including the renormalization effects are given. Sec.4 is devoted to the analysis of the quantum effects on the Kerr black hole. We find that the asymptotically flat region remains intact and stable. On the other hand, in a certain approximation at one loop order, it is shown that the inner geometry undergoes a considerable change no matter how small the quantum effects may be. Finally, brief discussion is given in Sec.5. Throughout this article, we adopt the sign convention in which the flat space-time metric in four dimensions is $\eta_{MN} = \mathrm{diag}(-1,1,1,1)$.

2 Dimensionally Reduced Einstein Gravity

In this section, we consider the dimensional reduction of the Einstein gravity with two commuting Killing vectors. By dropping the dependence on the direction of one isometry, and performing a dual transformation, we find that the theory is described by a $SL(2,\mathbf{R})/U(1)$ nonlinear σ-model coupled to gravity in three dimensions. Then we further reduce the theory to two dimensions. We shall follow the method adopted in [10], and deal with the case in which one Killing vector is time-like and the other is space-like.

We begin with the following vierbein in a triangular gauge,

$$E^A_{\ M} = \begin{pmatrix} \Delta^{-1/2}\, e^a_m & \Delta^{1/2} A_m \\ 0 & \Delta^{1/2} \end{pmatrix}, \tag{2.1}$$

where $M(=0-3)$ and $m(=1-3)$ refer to the space-time indices and $A(=0-3)$ and $a(=1-3)$ to those of its tangent space. Assuming that all the components are independent of the time-like coordinate, x^0, the Einstein-Hilbert action is reduced to

$$\begin{aligned} \frac{1}{\hbar} S_{EH} &= \frac{1}{\hbar\kappa}\int d^4x\ ER^{(4)}(E) \\ &= \frac{L}{l_p^2}\int d^3x\ e\left[\ R^{(3)}(e) + \frac{1}{4}\Delta^2 F_{mn}F^{mn} - \frac{1}{2}\gamma^{mn}\Delta^{-2}\partial_m\Delta\partial_n\Delta\right], \end{aligned} \tag{2.2}$$

where κ is given by $\kappa = G/c^3$, l_p is the Planck length, L is the "length" of x^0 direction, and E and e are det E^A_M and det e^a_m, respectively. F_{mn} is defined by $F_{mn} = (dA)_{mn} \equiv \partial_m A_n - \partial_n A_m$ and the indices are raised and lowered by the three metric, γ_{mn}, determined by the dreibein e^a_m.

The equations of motion derived from the above reduced action have a $SL(2,\mathbf{R})$ symmetry. Although it is not manifest in Eq.(2.2), we can obtain the action manifestly invariant under this symmetry by a dual transformation [1]. First, let us introduce an auxiliary field, C_{mn}, add a term to the Lagrangian, $\mathcal{L}$, as

$$\mathcal{L} \ \rightarrow\ \mathcal{L} + C^{mn}\left[F_{mn} - (dA)_{mn}\right], \tag{2.3}$$

and regard F_{mn} as an independent field. By integrating out C_{mn}, we get $F_{mn} = (dA)_{mn}$ and the original action. On the other hand, the integration of A_m leads to $\nabla_m C^{mn} = 0$ and hence C_{mn} can be written by a scalar field B as $C^{mn} = \frac{1}{2}\epsilon^{mnl}\partial_l B$, where ∇_m and ϵ_{mnl} are the covariant derivative operator and the volume element, respectively. Finally,

[1] The author would like to thank Y. Kazama for the discussion on this point.

by the further integration of F_{mn}, we get the following Lagrangian,

$$e\,\mathcal{L}^{(3)} \;=\; e\left[\, R^{(3)}(e) - \frac{1}{2}\gamma^{mn}\Delta^{-2}\left(\partial_m B\partial_n B + \partial_m\Delta\partial_n\Delta\right)\,\right], \tag{2.4}$$

and the relation between F_{mn} and B,

$$\Delta^2 F_{mn} \;=\; -\epsilon_{mnl}\,\partial^l B\,. \tag{2.5}$$

We can check that the model obtained in this way is actually equivalent on-shell to the original one. As intended, $\mathcal{L}^{(3)}$ has a $SL(2,\mathbf{R})$ symmetry,

$$Z \longrightarrow Z' \;=\; \frac{aZ+b}{cZ+d}\,; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{R})\,, \tag{2.6}$$

where $Z = B + i\Delta$. Z is related to the so-called Ernst potential $\mathcal{E}$ by

$$\mathcal{E} \;=\; i\bar{Z} \;=\; \Delta + iB\,. \tag{2.7}$$

Moreover we find that the model described by $\mathcal{L}^{(3)}$ is a $SL(2,\mathbf{R})/U(1)$ nonlinear σ-model (coupled to gravity), and this is a non-compact analog of a $O(3)$ ($\mathbf{C}P^1$) nonlinear σ-model. The analogy becomes obvious in the forms,

$$\begin{aligned} &-\frac{1}{2}\gamma^{mn}\Delta^{-2}\left(\partial_m B\partial_n B + \partial_m\Delta\partial_n\Delta\right) \\ &\quad = -\frac{1}{2}\gamma^{mn}\Delta^{-2}\partial_m\mathcal{E}\partial_n\mathcal{E} \;=\; -2\gamma^{mn}\frac{\partial_m w\partial_n\bar{w}}{(1-w\bar{w})^2} \;=\; -\frac{1}{2}\gamma^{mn}\,\eta^{ab}\partial_m v^a\partial_n v^b\,, \end{aligned} \tag{2.8}$$

where $(w-1)/(w+1) = \mathcal{E}$, and v^a is defined by $w = (iv^1 - v^2)/(1+v^0)$, $\eta_{ab}v^a v^b = -1$, and η_{ab} =diag$(-1,1,1)$ [25].

Now let us reduce the theory further to two dimensions. As in the previous case, we take the following form of the dreibein in a triangular gauge,

$$e^a_{\ m} \;=\; \begin{pmatrix} \lambda\delta^\alpha_\mu & \rho C_\mu \\ 0 & \rho \end{pmatrix}, \tag{2.9}$$

where $\mu,\alpha = 1,2$. Since C_μ has no physical degrees of freedom, we can set $C_\mu = 0$. Then by dropping the dependence on x^3, we obtain

$$\begin{aligned} \frac{1}{\hbar}S_{EH} \;&=\; \frac{V}{l_p^2}\int d^2x\; e\;\mathcal{L}^{(2)}\,, \\ e\;\mathcal{L}^{(2)} \;&=\; e\left[R^{(2)} - \frac{1}{2}\gamma^{\mu\nu}\Delta^{-2}\partial_\mu\mathcal{E}\partial_\nu\bar{\mathcal{E}}\right] \;=\; \rho\;\delta^{\mu\nu}\left[-2\partial_\mu\partial_\nu\ln\lambda - \frac{1}{2}\Delta^{-2}\partial_\mu\mathcal{E}\partial_\nu\bar{\mathcal{E}}\right], \end{aligned} \tag{2.10}$$

where V is the "volume" of (x^0, x^3) space-time. Note that, in the latter form, the indices are contracted effectively by the flat two dimensional metric $\delta_{\mu\nu}$. Thus in the following, it is understood that the indices are raised and lowered by the flat metric.

The independent equations of motion deduced from the above action are

$$\partial_\mu \partial^\mu \rho = 0 \tag{2.11}$$

$$\Delta \partial^\mu \left(\rho \partial_\mu \mathcal{E}\right) = \rho \ \partial_\mu \mathcal{E} \partial^\mu \mathcal{E} \tag{2.12}$$

$$\partial_\zeta \rho \partial_\zeta \ln \lambda - \frac{1}{2}\partial_\zeta^2 \rho = \frac{1}{4} \ \rho \ \Delta^{-2} \partial_\zeta \mathcal{E} \partial_\zeta \bar{\mathcal{E}} \, , \tag{2.13}$$

where $\zeta = x^1 + ix^2$. These equations are derived by the variations of $\mathcal{E}$ and $\gamma_{\mu\nu}$. The variation of ρ leads to a dependent equation. This is related to the fact that the degree of freedom of ρ is spurious. Indeed since ρ is a free field and there remains the choice of the conformal gauge in two dimensions preserving the form of $e^a{}_m$ in (2.9), we can identify ρ with one of the coordinates by some conformal transformation.

In the reduced theory to two dimensions, (2.5) leads to $A_{1,2} = 0$ and

$$\Delta^2 \partial_\zeta A \ \ = \ \ i\rho \partial_\zeta B \, . \tag{2.14}$$

Consequently, we have four basic equations, Eq.(2.11) $-$ Eq.(2.14). Eq.(2.12) for $\mathcal{E}$ is known as the Ernst equation and becomes integrable if we set ρ to be one of the coordinates. There exists vast knowledge of this equation. For detail, see [9, 10], [13].

As for the metric in four dimensions, in our parametrization we have

$$\begin{aligned} ds^2 \ &= \ \gamma_{MN} dx^M dx^N \ = \ \eta_{AB} E^A_{\ M} E^B_{\ N} \ dx^M dx^N \\ &= \ \Delta^{-1} \left[\lambda^2 \left((dx^1)^2 + (dx^2)^2\right) + \rho^2 (dx^3)^2\right] - \Delta \left(dx^0 + A \ dx^3\right)^2 . \end{aligned} \tag{2.15}$$

3 Renormalization of Noninear σ-model Part

In the previous section, we formulated the Einstein gravity with two commuting Killing vectors as a two dimensional $SL(2, \mathbf{R})/U(1)$ nonlinear σ-model coupled to gravity. In Sec.3, we consider the quantization of the nonlinear σ-model part which includes all the dynamical degrees of freedom. This means that we investigate the effects of the quantum fluctuations maintaining the symmetry of the isometries (independence of x^0 and x^3). Because the three dimensional gravity part $(\lambda\, , \rho$ etc.) has no physical degrees of freedom, we can expect that it does not make main contributions to the quantum effects. Thus we

left the quantization of this part as a future problem. In quantum theory, it is ambiguous which variables we should regard as fundamental to be quantized. The reasons we start our quantum analysis with this nonlinear σ-model are two folds. One is that the original hidden symmetry is manifest in this formulation. The other is that we can make use of the knowledge of the quantum theory of nonlinear σ-models developed in the literature. Due to this, the quantum analysis of our model is fairly simplified.

Since the fluctuations to x^0 and x^3 directions are ignored, such analysis is not enough to know the full quantum properties of Einstein gravity. In particular, we can say only a little about its statistical aspects. However, we have at present no consistent way to investigate the full quantum theory of Einstein gravity because of its nonrenormalizability and various difficulties. Our attitude here is a modest one. Although only a part of the quantum fluctuations can be incorporated, in this simplified setting we can carry out a consistent quantum analysis of Einstein gravity and extract some quantum effects on geometry. We believe that our analysis gives some insights into quantum aspects of general relativity. Indeed, it turns out that we can obtain the forms of the beta functions to any loop order and the renormalization effects on the classical solutions.

In order to respect the covariance of the target manifold, we rewrite the action of the nonlinear σ-model part by using its metric, $g_{ij}(\phi)$, and coordinates, ϕ^i,

$$\frac{1}{\hbar}S_{NL} \quad = \quad -\frac{1}{2e_0^2}\int d^2x\rho g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j\,, \tag{3.1}$$

where $e_0^2 = l_p^2/V$ is the coupling of the model. In string theory, V corresponds to the volume of the compactified space. On the contrary, in our context V is the "volume" of the real space-time, (x^0, x^3), and hence e_0^2 is an extremely small number, i.e., the model has a quite small coupling. The fluctuations depending only on x^1 and x^2 are constant modes with respect to the reduced directions, x^0 and x^3, and $e_0^2 \propto V^{-1}$ indicates that such fluctuations are suppressed by the "volume" of the constant direction. In the stationary axisymmetric case, which has the Kerr solution, the time x^0 runs from $-\infty$ to $+\infty$ and V tends to infinity. We do not know which value e_0^2 takes in such a case, but the coupling is still expected to be quite small.

In the coordinates $\phi^{1,2} = \Delta, B$, we have $g_{ij} = g\delta_{ij}$, $g = \Delta^{-2}$. In two dimensions, the curvature tensors are easily calculated through

$$R \quad = \quad -g^{-1}\delta^{ij}\partial_i\partial_j \ln g\,, \tag{3.2}$$

$$
\begin{aligned}
R_{ij} &= \frac{1}{2}Rg_{ij}\,, && (3.3)\\
R_{ijkl} &= \frac{1}{2}R\left(g_{ik}g_{jl}-g_{il}g_{jk}\right)\,. && (3.4)
\end{aligned}
$$

From the first equation above, we get

$$
R \;=\; -2 \;=\; \text{const.} \quad , \qquad (3.5)
$$

namely, $SL(2,\mathbf{R})/U(1)$ manifold has a negative constant scalar curvature. Although the above form of g_{ij} appears singular at $\Delta = 0$, this is just a coordinate singularity and we can get regular metric even there by some appropriate coordinate transformation.

Now let us consider the renormalization of the effective action of the model. As the functional measure, we take $\prod_{x,i}\sqrt{\det g_{ij}}d\phi^i(x)$. This is invariant under coordinate transformations of the target manifold, and respects the covariance. The only difference of our model from ordinary nonlinear σ-models is the existence of the factor ρ in (3.1). This factor behaves as a coordinate-dependent coupling like dilaton field in string theory. In the following, we assume $\rho(x) > 0$. The reality of the space-time metric requires just that $\rho(x)$ is real or pure imaginary (see (2.15)). In the case of negative or pure imaginary ρ, we have only to replace $\rho(x)$ with $|\rho(x)|$. With these in mind, we shall adopt the background field method and follow [23]. Thus our analysis does not depend on which background we shall take.

First, we expand the action around the background fields, φ^i, by normal coordinates,

$$
\begin{aligned}
-\frac{1}{\hbar}S_{NL}[\phi] \;=\; & -\frac{1}{\hbar}S_{NL}[\varphi] + \int d^2x\; T_0^{-1} g_{ij}(\varphi)\partial_\mu\varphi^i D^\mu\xi^j \\
& +\frac{1}{2}\int d^2x\; T_0^{-1}\Big[g_{ij}D_\mu\xi^i D^\mu\xi^j + R_{ik_1k_2j}\xi^{k_1}\xi^{k_2}\partial_\mu\varphi^i\partial^\mu\varphi^j \\
& +\frac{1}{3}D_{k_1}R_{ik_2k_3j}\xi^{k_1}\xi^{k_2}\xi^{k_3}\partial_\mu\varphi^i\partial^\mu\varphi^j + \frac{4}{3}R_{ik_1k_2k_3}\xi^{k_1}\xi^{k_2}D_\mu\xi^{k_3}\partial^\mu\varphi^i + \cdots\Big]\,,
\end{aligned}
\qquad (3.6)
$$

where ξ^i is the tangent vector to the geodesics around φ^i, $T_0(x) = e_0^2/\rho(x)$, and $D_\mu\xi^i = \partial_\mu\xi^i + \Gamma^i_{jk}\partial_\mu\varphi^k\xi^j$. D_k is the covariant derivative, e.g., $D_k\xi^i = \partial_k\xi^i + \Gamma^i_{jk}\xi^k$, and Γ^i_{jk} is the Christoffel symbol defined by g_{ij}. Next, we introduce the zweibein, $\hat{h}^p{}_i(\varphi,\rho)$, with respect to $\hat{g}_{ij}(\varphi,\rho) \equiv \rho g_{ij}(\varphi)$ and with the properties

$$
\hat{h}^p{}_i\,\hat{h}_{pj} = \hat{g}_{ij} \quad , \quad \hat{h}^j{}_p\,\hat{h}^q{}_j = \delta^q{}_p\,. \qquad (3.7)
$$

Here the indices for the target manifold, i, j, are raised and lowered by $\hat{g}_{ij}$ while those for its tangent space, p, q, by δ_{pq}. (Henceforth we denote the quantities with respect to $\hat{g}_{ij}$

by the hat $\hat{}$.) $\hat{h}^p{}_i$ is expressed by the zweibein, $h^p{}_j$, with respect to g_{ij} as $\hat{h}^p{}_i = \sqrt{\rho}\ h^p{}_i$. Then we define $\xi^p = h^p{}_j\ \xi^j$ and $\hat{\xi}^p = \hat{h}^p{}_i\xi^i = \sqrt{\rho}\xi^p$. Noting that $\hat{h}^p{}_i$ depends not only on φ but also on ρ, we have

$$\hat{h}^p{}_j D_\mu \xi^j \;=\; \hat{\mathcal{D}}_\mu\ \hat{\xi}^p \;\equiv\; \hat{D}_\mu \hat{\xi}^p - \frac{1}{2}\partial_\mu \ln\rho \cdot \hat{\xi}^p\,, \tag{3.8}$$

where $\hat{D}_\mu\hat{\xi}^p = \partial_\mu\hat{\xi}^p + \hat{A}^p{}_{q\mu}\hat{\xi}^q$, $\hat{A}^p{}_{q\mu} = \hat{\omega}^p{}_{qk}\partial_\mu\varphi^k$, and $\hat{\omega}^p{}_{qk} = \hat{h}^p{}_j\ (\partial_k\hat{h}^j{}_q + \hat{\Gamma}^j_{kl}\ \hat{h}^l{}_q)$. Here we have used $\hat{\Gamma}^i_{jk} = \Gamma^i_{jk}$. In terms of $\hat{\xi}^p$, the kinetic term has the canonical form,

$$\hat{g}_{ij}D_\mu\xi^i D^\mu\xi^j \;=\; \hat{\mathcal{D}}_\mu\hat{\xi}^p\hat{\mathcal{D}}^\mu\hat{\xi}_p \;=\; \partial_\mu\hat{\xi}^p\partial^\mu\hat{\xi}_p + \cdots\,. \tag{3.9}$$

In order to see how other terms are expressed by $\hat{\xi}^p$, we assign weight N to quantities with the property $\Phi^{(N)}(\Lambda g_{ij}) = \Lambda^N\Phi^{(N)}(g_{ij})$ where Λ is a constant. In each term in (3.6), the part without ξ^i and expressed by the geometrical quantities through g_{ij} has weight 1 because we are originally expanding $g_{ij}(\phi)\ \partial_\mu\phi\partial_\nu\phi$. Let us denote such a quantity by $\Phi^{(1)}(g_{ij})$. Since the derivatives of ρ with respect to φ^k vanish, i.e., $\partial_k\rho = 0$, it holds that

$$\rho\Phi^{(1)}(g_{ij}) \;=\; \rho\Phi^{(1)}(\hat{g}_{ij}/\rho) \;=\; \Phi^{(1)}(\hat{g}_{ij}) \;\equiv\; \hat{\Phi}^{(1)}\,. \tag{3.10}$$

For example, $\rho R_{ijkl}(g_{ij}) = R_{ijkl}(\hat{g}_{ij}) \equiv \hat{R}_{ijkl}$. From Eqs.(3.8) and (3.10), we obtain

$$\begin{aligned}
-\frac{1}{\hbar}S_{NL}[\phi] \;=\; & -\frac{1}{\hbar}S_{NL}[\varphi] + \int d^2x\ T_0^{-1}g_{ij}(\varphi)\partial_\mu\varphi^i D^\mu\xi^j \\
& + \frac{1}{2e_0^2}\int d^2x\,\Big[\ \hat{\mathcal{D}}_\mu\hat{\xi}^p\hat{\mathcal{D}}^\mu\hat{\xi}_p + \hat{R}_{ip_1p_2j}\hat{\xi}^{p_1}\hat{\xi}^{p_2}\partial_\mu\varphi^i\partial^\mu\varphi^j \\
& + \frac{1}{3}\hat{D}_{p_1}\hat{R}_{ip_2p_3j}\hat{\xi}^{p_1}\hat{\xi}^{p_2}\hat{\xi}^{p_3}\partial_\mu\varphi^i\partial^\mu\varphi^j + \frac{4}{3}\hat{R}_{ip_1p_2p_3}\hat{\xi}^{p_1}\hat{\xi}^{p_2}\hat{\mathcal{D}}_\mu\hat{\xi}^{p_3}\partial^\mu\varphi^i + \cdots\Big]\,,
\end{aligned} \tag{3.11}$$

where $\hat{R}_{pijk} = \hat{h}^l{}_p\hat{R}_{lijk}$ etc. . Therefore we find that the changes from the cases without ρ (i.e., ordinary nonlinear σ-models) are only (i) the replacement of all the quantities by those with the hats and (ii) the further replacement $\hat{D}_\mu\hat{\xi}^p \to \hat{\mathcal{D}}_\mu\hat{\xi}^p$. The term linear in ξ^i contributes to a field redefinition together with the source term omitted here. We shall drop this linear term because it is irrelevant to the following discussion. Since the transformations, $\phi^i \to \xi^i \to \xi^p$, are coordinate transformations on the manifold, the functional measure is invariant, while under the last transformation, $\xi^p \to \hat{\xi}^p$, the measure is changed into $\prod_{x,p}\rho^{-1}\ d\hat{\xi}^p(x)$. When the factor ρ^{-1} is raised into the action, it is proportional to the delta function. However, since we shall adopt dimensional regularization, it plays no role in the following calculations at least perturbatively [21, 22].

We now proceed to the loop calculations. As long as we are concerned with divergent parts, we can estimate the effects due to $\partial_\mu \ln\rho$ in $\hat{\mathcal{D}}_\mu \hat{\xi}^p$ to all orders. First, let us note that possible counter terms are scalars, and on dimensional grounds they are of dimension two and hence including two base-space derivatives. Second, at N-loop order, they have weight $-N+1$. Third, since $\hat{R} = -2/\rho$ and similar formulae to (3.2)$-$(3.4) are valid for the quantities with the hats, the covariant derivatives of the curvatures, $\hat{R}$, $\hat{R}_{ij}$, and $\hat{R}_{ijkl}$, vanish and any scalar without the base-space derivative, ∂_μ, is a function only of ρ. Therefore the counter terms at N-loop order has the factor T_0^{N-1}. For example at one and two loop orders, the possible counter terms including $\partial_\mu \ln\rho$ are proportional to $\partial_\mu \ln\rho\, \partial^\mu \ln\rho$ and $\partial_\mu \ln\rho\, \partial^\mu \ln\rho\, \hat{R}$, respectively. Consequently, we find the counter terms due to $\partial_\mu \ln\rho$ in $\hat{\mathcal{D}}_\mu \hat{\xi}^p$ to be of the form

$$\delta S_{NL}^{(\rho)} = -\frac{1}{4\pi\epsilon}\int d^2x \left(\sum_{N=1} b_N\, T_0^{N-1}\right) \partial_\mu \ln\rho\, \partial^\mu \ln\rho\,, \tag{3.12}$$

where we have adopted the minimal subtraction and the dimensional regularization, i.e., dim.$= 2 \to n$ and $\epsilon = n-2$. As for the infrared regularization, we have adopted a simple mass cutoff. Since the renormalization of the model is a problem concerned with short distances, the scheme of the infrared regularization may not be essential. b_N are numerical coefficients determined by explicit calculations. It is easy to check $b_1 = 1/2$. The existence of $\delta S_{NL}^{(\rho)}$ shows that we have to add an additional bare term, $-1/2\int d^2x U_0^{-1} \partial_\mu \ln\rho\, \partial^\mu \ln\rho$, in the action, where $U_0 = \mathcal{O}(e_0^2)$.

As we have already estimated the result from the change $\hat{D}_\mu \to \hat{\mathcal{D}}_\mu$, the remaining analysis of the divergent parts can be performed in a parallel way to ordinary nonlinear σ-models. Thus we immediately get other counter terms up to two loop order [22, 23],

$$\begin{aligned}
\delta S_{NL} &= \frac{1}{4\pi\epsilon}\int d^2x \left[\hat{R}_{ij} + \frac{e_0^2}{4\pi}\, \hat{R}_{iklm}\hat{R}_j{}^{klm}\right] \partial_\mu\varphi^i \partial^\mu \varphi^j \\
&= \frac{1}{4\pi\epsilon}\int d^2x \left[-1 + \frac{T_0(x)}{2\pi}\right] g_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j\,.
\end{aligned} \tag{3.13}$$

Moreover we can determin the form of the remaining counter terms to all orders. In a similar way to the previous argument, we find that any tensor with two lower indices of the target manifold which is made out of the metric, curvatures and covariant derivatives are propotional to g_{ij}, and that the remaining counter terms are of the form

$$\delta S_{NL} = \frac{1}{4\pi\epsilon}\int d^2x \left[\sum_{N=1} a_N T_0^{N-1}\right] g_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j\,, \tag{3.14}$$

where a_N are numerical coefficients determined by the explicit calculations. (3.13) implies $a_1 = -1$ and $a_2 = 1/(2\pi)$. Note that the sign of a_1 is opposite to usual cases of compact manifolds.

As the counter terms above are functions of $\rho(x)$, the model is not strictly renormalizable. It is, however, renormalizable in a more general sense in which the manifold of the classical action changes due to quantum effects [22]. Indeed, we can derive beta functions for the couplings, $T(x;\mu) \equiv e^2(x;\mu)/\rho(x)$ and $U(T(x;\mu))$, as in the usual renormalizable theories [22, 23]. They are given by

$$\beta_T(T) \equiv \mu\frac{\partial}{\partial\mu}T = -\frac{1}{2\pi}\sum_{N=1} N a_N^{(0)} T^{N+1}, \tag{3.15}$$

$$\beta_U(T) \equiv \mu\frac{\partial}{\partial\mu}U = \frac{1}{2\pi}U^2\sum_{N=1}\left(N b_{N+1}^{(0)} + \partial_T \ln U \cdot b_N^{(0)}\right) T^N, \tag{3.16}$$

where μ represents the renormalization point, and $a_N^{(0)}$ and $b_N^{(0)}$ are the lowest part of a_N and b_N in ϵ^{-1}, respectively. It is easy to integrate the above equations as the beta functions are just rational functions of $\rho(x)$.

4 Quantum Effects on the Kerr Geometry

In the previous section, we carried out the renormalization of the model and derived the beta functions of the couplings. In this section, we investigate the physical consequences of our quantum analysis. We are interested in global geometry of space-time, and the effects of the higher derivative terms in the effective action are expected to be small for long distances. Thus we shall focus on the quantum effects due to the quadraric derivative terms in the effective action. Let us here regard μ_0 as representing the energy scale of the classical theory of the reduced gravity. Then $e^2(x;\mu_0) = e^2(\mu_0)$ holds, and the independent equations of motion including the quantum effects become

$$\partial_\mu\partial^\mu\rho = 0, \tag{4.1}$$

$$\Delta\partial_\mu\left(T^{-1}\,\partial^\mu\mathcal{E}\right) = T^{-1}\,\partial_\mu\mathcal{E}\partial^\mu\mathcal{E}, \tag{4.2}$$

$$\partial_\zeta\rho\partial_\zeta\ln\lambda - \frac{1}{2}\partial_\zeta^2\rho = \frac{1}{4}e^2(\mu_0)\left(T^{-1}\Delta^{-2}\partial_\zeta\mathcal{E}\partial_\zeta\bar{\mathcal{E}} + U^{-1}\partial_\zeta\ln\rho\partial_\zeta\ln\rho\right), \tag{4.3}$$

$$\Delta^2\partial_\zeta A = i\rho\partial_\zeta B. \tag{4.4}$$

Here we adopt a particular choice of the conformal gauge in two dimensions represented by (x^1, x^2). As mentioned in Sec.2, we can identify $\rho(x)$ one of the coordinates since $\rho(x)$

is a free field. Thus introducing another free field, z, conjugate with ρ, we choose the gauge,

$$x^1 = \sigma\rho\,, \qquad x^2 = \sigma z\,, \tag{4.5}$$

and hence $\zeta = \sigma(\rho + iz)$, where σ is some constant with dimension of length. In our context, only the Planck length, l_p, is such a constant made out of the fundamental constants in the theory. Then we set $\sigma = l_p$.

Now we consider the quantum effects on the Kerr geometry as an interesting example. It has been proved that the Kerr geometry is the unique solution to the stationary axisymmetric Einstein gravity under certain physical conditions [26]. In the following, we set $t \equiv x^0$ and $\omega \equiv x^3$, and regard t and ω as the time and the azimuthal angle, respectively. We shall find that the asymptotically flat region does not undergo any quantum correction, namely, the asymptotic region is stable. Furthermore, in a certain approximation at one loop order, it is shown that the geometry inside the ergosphere changes considerably no matter how small the quantum effects may be.

The Kerr solution to Eqs.(2.12)$-$(2.14) are usually expressed by Boyer-Lindquest coordinates, r (the radial coordinate) and θ (the polar angle), given by

$$\begin{aligned} l_p\rho &= x^1 = \sqrt{r^2 - 2mr + a^2}\ \sin\theta\,, \\ l_p z &= x^2 = (r-m)\cos\theta\,, \end{aligned} \tag{4.6}$$

where m and a turn out to represents the mass and the angular momentum per unit mass of the Kerr black hole, respectively. In these coordinates, the Kerr solution is given by [13]

$$\begin{aligned} \mathcal{E} &= \Delta + i\ B\,, \\ &\qquad \Delta = \frac{D - a^2\sin^2\theta}{\Sigma}\,, \quad B = \frac{2ma\cos\theta}{\Sigma}\,, \end{aligned} \tag{4.7}$$

$$\lambda^2 = \frac{D - a^2\sin^2\theta}{D + (m^2 - a^2)\sin^2\theta}\,, \quad A = a\frac{2mr\sin^2\theta}{D - a^2\sin^2\theta}\,, \tag{4.8}$$

where $D = r^2 - 2mr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$. Then the line element is written as

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2mr}{\Sigma}\right)dt^2 + \Sigma\left(\frac{dr^2}{D} + d\theta^2\right) \\ &\quad - \frac{4mar}{\Sigma}\sin^2\theta\ d\omega\ dt + \left(r^2 + a^2 + \frac{2ma^2r}{\Sigma}\sin^2\theta\right)\sin^2\theta\ d\omega^2\,. \end{aligned} \tag{4.9}$$

The zero of Σ and those of D (i.e., $r = r_\pm \equiv m \pm \sqrt{m^2 - a^2}$) correspond to the locations of the curvature singularity and the horizons, respectively, while the outer zero of $D - a^2 \sin^2\theta$ (i.e., $r = r_e \equiv m + \sqrt{m^2 - a^2\cos^2\theta}$) represents the outer boundary of the ergosphere. The asymptotically flat region is described by r (or ρ) $\to \infty$. In this asymptotic region, we have $(dx^1)^2 + (dx^2)^2 \sim (dr)^2 + r^2\ (d\theta)^2$, and (x^1, x^2) represents the flat 2-plane. Notice that ρ tends to vanish as $r \to r_\pm$ or $\sin\theta \to 0$.

Since the beta functions $\beta_T(T)$ and $\beta_U(T)$ are expanded by the power series of $T(x;\mu_0)$ $= e^2(\mu_0)/\rho(x)$, the perturbation is valid except for the small neighborhoods of order l_p of the axis of the rotation, $\sin\theta = 0$, and the horizons, $r = r_\pm$. This means that the quantum fluctuations become large there. It is obvious that in the asymptotic region the beta function vanishes. Therefore there is no quantum corrections due to the running couplings in that region and the flat region remains stable.

In order to further study the physical consequences of our analysis, we have to solve the equations (4.2)$-$(4.4). The change due to the term $U^{-1}\partial_\zeta \ln\rho\partial_{\bar\zeta}\ln\rho$ in (4.3) can be absorbed into a factor of λ. Let us define $f(\rho;\mu)$ and λ_T by $\lambda = f(\rho;\mu)\lambda_T$ and $f(\rho;\mu) \to 1$ as $\rho \to \infty$. Then taking into account $l_p\rho = x^1$, we find that $f(\rho;\mu)$ and λ_T are given by

$$f(\rho;\mu) \;=\; \exp\left(\frac{1}{4}e^2(\mu_0)\int_\rho^\infty d\rho'\ \rho'^{-2}U^{-1}(T(\rho';\mu))\right)\,, \tag{4.10}$$

$$\partial_\zeta\rho\partial_{\bar\zeta}\ln\lambda_T - \frac{1}{2}\partial_\zeta^2\rho \;=\; \frac{1}{4}e^2(\mu_0)T^{-1}\Delta^{-2}\partial_\zeta\mathcal{E}\partial_{\bar\zeta}\bar{\mathcal{E}}\,. \tag{4.11}$$

The equation for λ_T is of the same form as the classical equations for λ, (2.13), up to the replacement ρ with T^{-1}. At a generic order, however, remaining equations are quite complicated. Thus, henceforth, we focus on one loop order. At this order, we have $T^{-1}(x;\mu) = T^{-1}(x;\mu_0) - (1/2\pi)\ln(\mu/\mu_0)$, and we can get the solution to (4.2) and (4.11) from the classical one by the replacements of ρ and λ with $\rho - e^2(\mu_0)/2\pi\cdot\ln(\mu/\mu_0)$ and λ_T. Unfortunately, by this replacements the last equation (4.4) comes not to meet the integrability condition. Therefore we shall resort to further approximation. Here we consider the deviation from the classical solution in the neighborhood of $\rho(x) = \rho_0$, and approximate $T^{-1}(x;\mu)$ by

$$e^2(\mu_0)T^{-1}(x;\mu) \;=\; \rho(x)\left\{1 - \frac{1}{2\pi}e^2(\mu_0)\rho^{-1}(x)\ln(\mu/\mu_0)\right\} \;\sim\; \alpha(\rho_0)\rho(x)\,, \tag{4.12}$$

where $\alpha(\rho_0) = 1 - e^2(\mu_0)/2\pi\cdot\rho_0^{-1}\ln(\mu/\mu_0)$ =const. , and it tends to 1 as $\mu \to \mu_0$. This approximation is valid in the region where $\rho(x) >> 1$, because $\partial_{x^1}\rho^{-1} = -\ l_p^{-1}\rho^{-2}$ and

$\partial_{x^2}\rho^{-1} = 0$. In this approximation, (4.2) is the same as the classical one and the difference between (2.13) and (4.11) is only the exponents of λ and λ_T. Thus all the quantum effects is represented by the change of λ, and it is given by

$$\lambda^2 = f^2(\rho_0)(F_1/F_2)^{\alpha(\rho_0)}, \tag{4.13}$$

where $F_1 = D - a^2 \sin^2\theta$ and $F_2 = D + (m^2 - a^2)\sin^2\theta$. Therefore we find that in this approximation the geometry becomes

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2mr}{\Sigma}\right) dt^2 + f^2(\rho_0)\left(\frac{F_1}{F_2}\right)^{\alpha(\rho_0)-1} \Sigma \left(\frac{dr^2}{D} + d\theta^2\right) \\ &\quad - \frac{4mar}{\Sigma}\sin^2\theta \, d\omega \, dt + \left(r^2 + a^2 + \frac{2ma^2 r}{\Sigma}\sin^2\theta\right)\sin^2\theta \, d\omega^2 . \end{aligned} \tag{4.14}$$

From the above expression, we find that the additional zeros and singularities appear in the metric where F_1 or F_2 vanishes. The conformal properties of the geometry is very much affected by them. Moreover we see that these singularities develop curvature singularities. For example, let us consider one of the curvature invariants defined by $R_{0303} \equiv E^K_{\ 0} E^L_{\ 3} E^M_{\ 0} E^N_{\ 3} R^{(4)}_{KLMN}$. In our parametrization, it takes the form $R_{0303} = \lambda^{-2} F_3(\Delta, B, \rho)$, where F_3 is a certain function of Δ, B and ρ. Since, in the case of the Kerr geometry (i.e., $\alpha(\rho_0) = 1$), it becomes singular only at $\Sigma = 0$, R_{0303} comes to diverge at the zeros of F_1 or F_2 unless $\alpha(\rho_0) = 1$. Note that the condition $\rho(x) >> 1$ holds even there except for the vicinity of the axis of rotation as long as m and a are large enough compared with the Planck scale, and that the outest additional zeros or singularities occur at the outer boundary of the ergosphere, $r = r_e$. We need further investigation in order to know whether or not these singularities are true. However, our result indicates that the geometry inside the ergosphere, where unusual phenomena can take place, is changed considerably due to the quantum effects. This is the case no matter how small they may be, namely, as long as $\alpha(\rho_0) \neq 1$.

5 Discussion

In this article, we studied the quantum theory of the Einstein gravity with one time-like and one space-like Killing vector formulated as a $SL(2,\mathbf{R})/U(1)$ nonlinear σ-model. We showed that the quantum analysis of this model can be carried out in a parallel way to ordinary nonlinear σ-models in spite of the existence of an unusual coupling. This means that it is possible to investigate consistently the quantum aspects of Einstein gravity in

our limited case. In consequence, the forms of the beta functions were determined to all orders up to numerical coefficients. As an explicit example, we considered the quantum effects on the Kerr geometry. Then we found that the asymptotically flat region undergoes no quantum effects and remains stable. It is also discussed that the inner geometry of the Kerr black hole is changed considerably. These contrast with other quantum approaches to quantum properties of Einstein gravity, in which Minkowski space-time becomes unstable, and/or the solution different much from the classical one is discarded because of the validity of the perturbations [1]-[6].

It is obvious that we can deal with the case with two space-like Killing vectors in the same way, in which colliding wave solutions are known. In addition, the extension to the Einstein-Maxwell system is straightforward, because, when dimensionally reduced, this system is also formulated as a nonlinear σ-model coupled to gravity as mentioned in Introduction.

Admittedly, our analysis is incomplete to understand the full quantum properties of Einstein gravity. We can say nothing about the effects of the truncated degrees of freedom. Even after the dimensional reduction, the gravitational part remains to be quantized. We should also study the effects of the higher derivative terms in the effective action. In order to investigate the statistical aspects of Einstein gravity, we have to develop some other approaches. These are beyond the scope of this article and left as future problems. Since we have seen that Einstein's theory is formulated as a nonlinear σ-model already in the reduction to three dimensions, it may be interesting to consider the application of three dimensional nonlinear σ-models.

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